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A Three – Step – One Mapping Iterative Scheme for Multi-Valued Maps in W-Hyperbolic Space

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Abstract. In this paper , firstly we introduce A Three – Step – One Mapping Iterative scheme for multivalued mapping in W-Hyperbolic space secondly , we prove a convergence result of A Three – Step – One Mapping Iterative Scheme for multivalued nonexpansive mapping in the same space also established the convergence results when use Condition (I) and by using Hemicompact mapping in uniformly W-Hyperbolic space.

Keywords. A Three – Step – One Mapping Iterative Scheme , Multi-valued Nonexpansive mapping , uniformly Hyperbolic Space , Condition

1 - Introduction

In 2009 , Shahzad and Zegeye [1] introdused the convergence result of Ishikawa iteration for quasi nonexpansive multivalued maps in uniformly convex space as follows , for any $x_1 \in B$

$$x_{n+1} = (1-\tau_n)x_n + \tau_n \ell_n$$
$$y_n = (1-\mu_n)x_n + \mu_n t_n \ ; \forall n \geq 0$$

Where $\ell_n \in Ly_n , t_n \in Lx_n$, $\{\mu_n\}$ and$\{\tau_n\}$ are sequences in [0,1].

In 2010, Khan et al . [2] introdused weak and strong convergence result of a one step iteration for two multivalued nonexpansive mappings .

Also in 2011, Abbas et al . [3] established the convergence result of a new one step iteration for two multivalued nonexpansive mappings in uniformly convex Banach space as follows , for any $x_1 \in B$

$$x_{n+1} = \mu_n x_n + \tau_n t_n + \sigma_n \ell_n \ ; \forall n \geq 0$$

Where $\ell_n \in Lx_n , t_n \in Kx_n$, $\{\mu_n\},\{\tau_n\}$ and$\{\sigma_n\}$ are sequences in (0,1). In 2018, Birol G .and Sezgin A. [4] established the convergence result of a one step iteration for two multivalued nonexpansive mappings in uniformly convex W- Hyperbolic space as follows , for any $x_1 \in B$

$$x_{n+1} = W(\ell_n , W\left(x_n , t_n , \frac{\tau_n}{1-\mu_n}\right), \mu_n) \ ; \forall n \geq 0$$

Where $\ell_n \in Lx_n , t_n \in Kx_n$, $\{\mu_n\}$ and $\{\tau_n\}$ are sequences in(0,1).

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2 −Preliminaries

Definition (2.1)[5] : A metric space (E, d) is said to be hyperbolic space if there exist a map $W: E^2 \times [0,1] \to E$ satisfying :

$W1 - d(t, W(x, y, \mu)) \leq (1-\mu)d(t,x) + \mu\, d(t,y)$

$W2 - d(W(x, y, \mu), W(x, y, \delta)) = |\mu - \delta|\, d(x-y)$

$W3 - W(x, y, \mu) = W(y, x, (1-\mu))$

$W4 - d(W(x, c, \mu), W(y, e, \mu)) \leq (1-\mu)d(x,y) + \mu\, d(c,e)$

$\forall\, x, y, c, e \in E$ and $\mu, \delta \in [0,1]$.

Definition (2.2)[2]: Let A be a non−empty and convex subset of a hyperbolic space B . A mapping $L: A \to 2^A$ is said to be

(i) nonexpansive if $H(Lx, Ly) \leq d(x,y)$ ... (1)
$\forall\, x, y \in A$.

(ii) quasi nonexpansive if $F(L) \neq \emptyset$ and $d(Lx, \rho) \leq d(x, \rho)$ ... (2)
$\forall\, x \in A \text{ and } \rho \in F(L)$.

Definition (2.3) : Let A be a non−empty and convex subset of a hyperbolic space B and $L: A \to 2^A$ is a mapping for any $x_1 \in A$, the sequence $\{x_n\}$ define by

$$x_{n+1} = W(g_n),\ g_n \in Ly_n$$
$$y_n = W(\ell_n, t_n, \mu_n),\ \ell_n \in Lx_n \text{ and } t_n \in Lz_n$$
$$z_n = W(x_n, \ell_n, \tau_n),\ \ell_n \in Lx_n\ ; \forall n \geq 0 \quad \ldots (3)$$

where $\{\mu_n\}$ and $\{\tau_n\}$ are sequences in (0,1).

Definition (2.4) [6]: A hyperbolic space (E, d, W) is said to be uniformly convex if there exist $\sigma \in\,]0,1]$ such that

$d(x, \vartheta) \leq \gamma$, $d(y, \vartheta) \leq \gamma$ and $d(x,y) \geq \epsilon\gamma$ then

$$d\left(W\left(x, y, \frac{1}{2}\right), \vartheta\right) \leq (1-\sigma)\gamma$$

$\forall\, \vartheta, x, y \in E$, $\epsilon \in (0,2]$ and $\gamma > 0$

Lemma(2.5), [7] : Let B be a uniformly convex hyperbolic . Let $x \in B$ and $\{\mu_n\}$ be a sequence in $[f, h]$ for some $f, h \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in B such that $\lim_{n\to\infty} \sup d(x_n, x) \leq \varphi$, $\lim_{n\to\infty} \sup d(y_n, x) \leq \varphi$ and

$\lim_{n\to\infty} d(W(x_n, y_n, \mu_n), x) = \varphi$ for some $\varphi \geq 0$

then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Definition (2.6)[8] : Let A be a non−empty and convex subset of a hyperbolic space B . A mapping $L: A \to 2^A$ is said to be

(i) Satisfy condition (I) if $\exists$ a nondecreasing function $k: [0,\infty[\, \to [0,\infty[$ with $k(0) = 0$, $k(\alpha) > 0$ for $\alpha \in (0,\infty)$ such that

$d(x, Lx) \geq k(d(x, F(L)))$, $\forall x \in A$.

(ii) Hemicompact. If for any sequence $\{x_n\}$ in $A \ni \lim_{n\to\infty} d(x_n, Lx_n) = 0$, $\exists$ a subsequence $\{x_{n_t}\}$ of $\{x_n\}$ such that $x_{n_t} \to \rho$ as $\tau \to \infty$ for some $\rho \in A$.

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3 − Main Theorem

Theorem(3.1):

Let B is a hyperbolic space , A is a nonempty and convex subset of B and $L: A \to C(A)$ is a quasi nonexpansive mapping . Let $\{x_n\}$ define by condition (3) with μ_n *and* $\tau_n \in (0,1)$. If $\rho \in F(L)$, then $\lim_{n\to\infty} d(x_n, \rho)$ exists , $\forall \rho \in F(L)$.

Proof: Let $\rho \in F(L)$, from condition (2) and(3) , we get

$$d(z_n, \rho) = d(W(x_n, , \ell_n, \tau_n), \rho), \ell_n \in Lx_n$$
$$\leq (1 - \tau_n)d(x_n, \rho) + \tau_n d(\ell_n, \rho)$$
$$\leq (1 - \tau_n)d(x_n, \rho) + \tau_n H(Lx_n, \{\rho\})$$
$$\leq (1 - \tau_n)d(x_n, \rho) + \tau_n H(Lx_n, L\rho)$$
$$\leq (1 - \tau_n)d(x_n, \rho) + \tau_n d(x_n, \rho)$$
$$= d(x_n, \rho) \quad \text{... (4)}$$

From condition (2) , (3) and (4) , we get $d(y_n, \rho) = d(W(\ell_n, t_n, \mu_n), \rho)$, $\ell_n \in Lx_n$ *and* $t_n \in Lz_n$

$$\leq (1 - \mu_n)d(\ell_n, \rho) + \mu_n d(t_n, \rho)$$
$$\leq (1 - \mu_n)H(Lx_n, \{\rho\}) + \mu_n H(Lz_n, \{\rho\})$$
$$\leq (1 - \mu_n)H(Lx_n, \rho) + \mu_n H(Lz_n, \rho)$$
$$\leq (1 - \mu_n)d(x_n, \rho) + \mu_n d(z_n, \rho)$$
$$\leq (1 - \mu_n)d(x_n, \rho) + \mu_n d(x_n, \rho)$$
$$= d(x_n, \rho) \quad \text{... (5)}$$

From condition (2) , (3)(4), and (5) , we get

$$d(x_{n+1}, \rho) = d(W(g_n), \rho), \; g_n \in Ly_n$$
$$\leq H(Ly_n, \{\rho\})$$
$$\leq H(Ly_n, L\rho)$$
$$\leq d(y_n, \rho)$$
$$\leq d(x_n, \rho) \quad \text{... (6)}$$

Then $\{d(x_n, \rho)\}$ is bounded and non- increasing . Hence $\lim_{n\to\infty} d(x_n, \rho)$ exists , $\forall \rho \in F(L)$.

Theorem(3.2):

Let B , A, L and $\{x_n\}$ be as in theorem(3.1), where μ_n and τ_n are sequences in $[f, h]$ for some f, h *with* $0 < a \leq b < 1$. Then $F(L) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Lx_n, x_n) = 0$.

Proof: Suppose $F(L) \neq \emptyset$ andLet $\rho \in F(L)$, then $\lim_{n\to\infty} d(x_n, \rho)$ exists and $\{x_n\}$ is bounded .Put

$$\lim_{n\to\infty} d(x_n, \rho) = \varphi. \text{ for some } \varphi > 0 \quad \text{... (7)}$$

From condition (4) and (7), we get

$$\lim_{n\to\infty} \sup d(W(x_n, \ell x_n, \tau_n), \rho) = \lim_{n\to\infty} \sup d(z_n, \rho)$$
$$\leq \lim_{n\to\infty} \sup d(x_n, \rho) = \varphi \quad \text{... (8)}$$
$$\lim_{n\to\infty} \sup d(W(\ell x_n, t_n, \mu_n), \rho) = \lim_{n\to\infty} \sup d(y_n, \rho)$$
$$\leq \lim_{n\to\infty} \sup d(x_n, \rho) = \varphi \quad \text{... (9)}$$

From condition (2), (3)and (4), we get

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$$
\begin{aligned}
d(x_{n+1},\rho) &= d(W(g_n),\rho),\ g_n \in Ly_n \\
&\le H(Ly_n,\{\rho\}) \\
&\le H(Ly_n,L\rho) \\
&\le d(y_n,\rho) \\
&= d(W(\ell_n,t_n,\mu_n),\rho),\ \ell_n \in Lx_n \text{ and } t_n \in Lz_n \\
&\le (1-\mu_n)d(\ell_n,\rho)+\mu_n d(t_n,\rho) \\
&\le (1-\mu_n)H(L\,x_n,\{\rho\})+\mu_n H(L\,z_n,\{\rho\}) \\
&\le (1-\mu_n)H(L\,x_n,\rho)+\mu_n H(L\,z_n,\rho) \\
&\le (1-\mu_n)d(x_n,\rho)+\mu_n d(z_n,\rho) \\
&\le d(x_n,\rho)-\mu_n d(x_n,\rho)+\mu_n d(z_n,\rho)
\end{aligned}
$$

This implies that

$$\frac{d(x_{n+1},\rho)-d(x_n,\rho)}{\mu_n} \le d(z_n,\rho)-d(x_n,\rho)$$

So

$$
\begin{aligned}
d(x_{n+1},\rho)-d(x_n,\rho) &\le \frac{d(x_{n+1},\rho)-d(x_n,\rho)}{\mu_n} \\
&\le d(z_n,\rho)-d(x_n,\rho)
\end{aligned}
$$

Then $d(x_{n+1},\rho) \le d(z_n,\rho)$

Therefore

$$\varphi \le \liminf_{n\to\infty} d(z_n,\rho)\,. \qquad \ldots(10)$$

From condition (8) and (10) , we get

$$\varphi = \lim_{n\to\infty} d(z_n,\rho)$$

$$\varphi = \lim_{n\to\infty} d(W(x_n,\ell\, x_n,\tau_n),\rho) \qquad \ldots(11)$$

From condition (7) , (9), (11)and lemma (2.5),we get

$$\lim_{n\to\infty} d(\ell\, x_n,x_n) \le \lim_{n\to\infty} d(Lx_n,x_n) = 0.$$

Theorem(3.3)

Let B , A, L and $\{x_n\}$ be as in theorem(3.2) with $F(L) \ne \emptyset$ if L satisfies condition (I), then $\lim_{n\to\infty} \|x_n-\rho\| = 0\,, \rho \in F(L)$.

Proof:

From theorem (3.1), implies that $\lim_{n\to\infty} d(x_n,\rho)$ exists, $\forall\ \rho \in F(L)$ and so $\lim_{n\to\infty} d\big(x_n,F(L)\big)$ exists . Assume that $\lim_{n\to\infty} d(x_n,\rho) = \varphi$ for some $\varphi > 0$. From condition (I) , $F(L) \ne \emptyset$ and theorem(3.2), we get $\lim_{n\to\infty} k(d(x,F(L))) \le \lim_{n\to\infty} d(x_n,Lx_n) = 0$ implies that

$$\lim_{n\to\infty} k(d(x,F(L))) = 0 \qquad \ldots(12)$$

Since k is a nondecreasing function and from condition (12), we get $\lim_{n\to\infty} (d(x,F(L))) = 0$. Thus there are subsequences $\{x_{n_t}\}$ of $\{x_n\}$ and $\{\rho_t\} \subset F$ such that $d(x_{n_t},\rho_t) < \frac{1}{2^t}$

From theorem (3.1) , $\forall t > 0$, we get

$$d(x_{n_{t+1}},\rho_t) \le d(x_{n_t},\rho_t) < \frac{1}{2^t}$$

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Also , we get $\{\rho_t\}$ is a Caushy sequence in A and converges to $\rho \in A$ by

$$d(\rho_{t+1`}, \rho_t) \leq d\left(\rho_{t+1`}, x_{n_{t+1}}\right) + d(x_{n_{t+1}}, \rho_t)$$
$$< \frac{1}{2^{t+1}} + \frac{1}{2^t}$$
$$< \frac{1}{2^{t-1}}$$

Since $d(\rho_t, L\rho) \leq H(L\rho_t, L\rho) \leq d(\rho, \rho_t)$and $\rho_t \to \rho \; as \; \tau \to \infty$, it follows that $d(\rho, L\rho) = 0$, which implies that $\rho \in L\rho$.

Theorem(3.3)

Let B , A, L and $\{x_n\}$ be as in theorem(3.2) with $F(L) \neq \emptyset$ if L is hemicompact , then $\lim_{n\to\infty} \|x_n - \rho\| = 0$, $\rho \in F(L)$.

proof : From theorem(3.2) , we get $\lim_{n\to\infty} d(x_n, Lx_n) = 0$ and L is hemicompact , there is a subsequence $\{x_{n_t}\}$ of $\{x_n\}$ such that $x_{n_t} \to \rho$ as $\tau \to \infty$ for some $\rho \in A$. Since L is a nonexpansive mapping ,

$$d(\rho, L\rho) \leq d\left(\rho, x_{n_t}\right) + d\left(x_{n_t}, Lx_{n_t}\right) + H\left(Lx_{n_t}, L\rho\right)$$
$$\leq 2d\left(x_{n_t}, \rho\right) + d\left(x_{n_t}, Lx_{n_t}\right) \quad \to 0 \qquad as \;\; \tau \to \infty \, .$$

this implies that $\rho \in L\rho$.

4 −Conclusions

A Three – Step – One Mapping Iterative scheme for multivalued mapping has been introduced in W-Hyperbolic and the convergence result for A Three – Step – One Mapping Iterative Scheme has been proved when used multivalued nonexpansive mapping in Hyperbolic space also established the convergence results has been established in uniformly Hyperbolic space when use Condition (I) and use Hemicompact mapping .

5 − Suggestion

1 − we can use multivalued Lipschizian mapping to established the convergence results .

2 − we can use another space to establish the convergence results of same iteration .

3 − we can use another iteration to established the convergence results .

References

[1] Shahazad N. and Zegeye H. , On Mann an Ishikawa iteration schemes for multivalued mapping in Banach spaces, Nonlinear Anal. 71 (2009), no. 3-4 ,838-844.

[2] Khan S.H. , Abbas M. and Rhoades B.E . ,A new one – step iterative scheme for approximating comman fixed points of two multivalued nonexpansive mappings , Rend, Circ. Mat. 59(2010),149-157 .

[3] Abass M. , Khan S.H. , Khan A. R. and Agarwal R.P. ,Common fixed points of two multivalued nonexpansive mappings by one –step iterative scheme , Appl. Math. Lett. 24 (2011), 97-102 .

[4] Karahan I . , Convergence of SP iteration scheme for three mulivalued mappings in Hyperbolic space , Journal of Comp. Anal. and Appl. 24(5) (2018):815-827 .

[5] Kohlenbach U. , Some logical metatheorems with applications in functional analysis , Tra. Amer. Math. Soc. 357 (2005) 89-128 .

[6] Gürsoy F. , Karakaya V. , Some convergence and stability results for two new kirk type hybrid fixed point iterative algorithms. Journal of Function spaces doi: 10.1155/2014/684191.
[7] Shimizu T. and Takahashi W. ,Fixed points of multivalued mappings in certain convex metric spaces .Topol Methods Nonlinear Anal. 8,197-203,(1996).
[8] Khan A. R. , Fukhar – ud – H . and Khan M . A. A. ,An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces .Fixed Point Theory and Applications, 2012 (54) .