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Four classical methods in solving inequalities

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Abstract. Inequality has always been important in algebra, and so many scientists have come up with many ideas to prove inequalities.In this article we want to examine 4 classic methods of proving inequality called Jensen, S.O.S, Lagrange method and Chinese Dumbass

Keywords. Jensen's inequality , PQR , SOS , Chinese Dumbass

1 Introduction

Inequalities can be found in various parts of mathematics. Inequalities in different fields of mathematics have very fascinating properties and various applications. As it is reduced into a suitable algebraic form before continuing any further with any known inequality, an inequality becomes simple to control [5].The first method we examine in this article is Jensen inequality.We prove Jensen's Inequality by an inductive argument on the number of points[2].The second part of the articleis centered around two very powerful tools for solving polynomialtype inequalities: the $S.O.S$ technique and Chinese Dumbass Notation[4].And the last method, the Lagrange multipliers method.Lagrange multipliers offer a way to find the extrema of an objective function $f(x_1, x_2, \dots, x_n)$ subject to a constraint function $g(x_1, x_2, \dots, x_n) = 0$. [3]

2 Jensen inequality

Definition 2.1 *Let $C \in R^N$ be non-empty and convex and let $f: C \to R$.*

1. (a) f is concave iff for any $a, b \in C$ and any $\theta \in [0,1]$:

$$f(\theta a + (1-\theta)b) \geq \theta f(a) + (1-\theta)f(b)$$

(b) f is strictly concave iff for any $a, b \in C$ and any $\theta \in (0,1)$, the above inequality is strict.

2. (a) f is convex iff for any $a, b \in C$ and any $\theta \in [0,1]$:

$$f(\theta a + (1-\theta)b) \geq \theta f(a) + (1-\theta)f(b)$$

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(b) f is strictly convex iff for any $a, b \in C$ and any $\theta \in (0,1)$, the above inequality is strict.[1]

Theorem 2.2 *(a) Let f be a convex function on the interval I. Then*

$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n)$
where
$x_1, x_2, \cdots, x_n \in I$, $\lambda_1, \lambda_2, \cdots, \lambda_n \in [0,1]$
(b) Let f be a convex function on the interval I. Then

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \geq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n)$$

where
$x_1, x_2, \cdots, x_n \in I$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0,1]$

Proof. We prove Jensen's Inequality by an inductive argument on the number of points. When $n = 2$, the inequality follows from the definition of convexity. Assuming that it is true for $n - 1$ many points, we show its validity for n many points. Let $\lambda_1, \cdots, \lambda_n \in (0,1)$, $\sum \lambda_i = 1$, and

$$y = \sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} x_k \in [x_1, x_{n-1}]$$

Using first the definition of convexity and then the induction hypothesis

$$f(\lambda_1 x_1 + \ldots . + \lambda_n x_n) = f((1 - \lambda_n) y + \lambda_n x_n) \leq (1 - \lambda_n) f(y) + \lambda_n f(x_n)$$

$$= (1 - \lambda_n) f(\sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} x_k) + \lambda_n f(x_n)$$

$$\leq (1 - \lambda_n)(\sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} f(x_k)) + \lambda_n f(x_n) = \sum_{i=1}^{n} \lambda_i f(x_i)$$

The case when $\lambda_k = 1$ for some k is trivial. On the other hand, when some λ_k is 0, the inequality reduces to one with fewer λ_k's, and its validity comes from the induction hypothesis.When f is strictly convex and $\lambda_k \in (0,1)$ for all k, it follows straightly from definition that the strict inequality sign in Jensen's inequality holds when $n = 2, x_1 \neq x_2$. In general, let us assume that the strictly inequality sign holds when $x_1, \cdots, x_{n-1}$ are distinct and prove it when $x_1, \ldots, x_n$ are not all equal. For, when all $x_1, \ldots, x_n$ are distinct, the second $\leq$ in the above inequalities becomes $<$ due to the induction hypothesis and hence the strict inequality holds for n. When some x_k's are equal, we can group the expression $\sum_{i=1}^{n} \lambda_i x_i$ into $\sum_{i=1}^{m} \mu_i y_i$ where all y_k's are distinct and m is less than n. In this case the desired result comes from the induction hypothesis.

The case of equality becomes trivial when some λ_k equals to 1. When $\lambda_k = 0$, for some k, the inequality is the same as we remove all terms containing λ_k, $k \in I_2$.from both sides. The rest λ_k, $k \in I_1$, are in (0,1), so the desired conclusion follows as before.[2]

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Example 2.3 *if* $a, b, c \in R^+$ *prove that :* $\frac{9}{a+b+c} \leq \sum \frac{2}{a+b}$ *[6]*

Proof. According to jensen's inequality for the function $f(x) = \frac{1}{x}$ and $f: R \rightarrow R$ for positive values x we know that

$$\frac{\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}}{3} \geq \frac{1}{\frac{2}{3}a+b+c} \rightarrow \sum \frac{1}{a+b} \geq \frac{3}{\frac{2}{3}\sum a}$$

And inequality is proved

3 $S.O.S$ Method

Theorem 3.1 *The basic idea is to write an inequality in the form*
$S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq 0$.

If it so happens that S_a, S_b and S_c are nonnegative, then we are done. For sharper inequalities this may not be the case; however, most of the time we can still finish using some minor adjustments. Some of the nicer estimates are provided below. They are drawn from [4]

Example 3.2
if a, b, c are positive real number we know

$$1.\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} - \frac{3}{2} = \sum \frac{(a-b)^2}{2(a+c)(b+c)}$$

$2 .\ \sqrt{2(a^2+b^2)} - (a+b) = \frac{(a-b)^2}{a+b\sqrt{2(a^2+b^2)}}$

$3 .\ n\sum_{i=1}^{n} a_i^2 - (\sum_{i=1}^{n} a_i) = \left(\sum_{i<j} (a_i - a_j)^2\right)$

$4 .\ (a+b)(b+c)(c+a) - 8abc = \sum a(b-c)^2$

Example 3.3 *Let* x, y, z *be three positive reals such that* $xyz \geq 1$. *Prove that[6]*

$$\frac{x^5-x^2}{x^5+y^2+z^2} + \frac{y^5-y^2}{x^2+y^5+z^2} + \frac{z^5-z^2}{x^2+y^2+z^5} \geq 0.$$

Proof. :

$$\frac{x^5-x^2}{x^5+y^2+z^2} - \frac{x^5-x^2}{x^3(x^2+y^2+z^2)} = \frac{(x^3-1)^2x^2(y^2+z^2)}{x^3(x^2+y^2+z^2)(x^5+y^2+z^2)} \geq 0$$

Therefore

$$\sum \frac{x^5-x^2}{x^5+y^2+z^2} \geq \sum \frac{x^5-x^2}{x^3(x^2+y^2+z^2)} = \frac{1}{x^2+y^2+z^2}\sum \left(x^2 - \frac{1}{x}\right) \geq \frac{1}{x^2+y^2+z^2}\sum (x^2 - yz) \geq 0$$

4 Chinese Dumbass

Theorem 4.1 *Chinese Dumbass Notation is a convenient way of presenting a polynomial inequality, making it significantly easier to find the suitable applications.The idea is that the*

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coefficients of an inequality of the form $P(a,b,c) \geq 0$ are held in a triangle of side length deg $P+1$, where the coefficient of the $a^x b^y c^z$ term are located at the point with barycentric coordinates $(x:y:z)$. Explicitly for, say, the third degree:

$$\begin{array}{ccccccc} & & & [a^3] & & & \\ & & [a^2b] & & [a^2c] & & \\ & [ab^2] & & [abc] & & [ac^2] & \\ [b^3] & & [b^2c] & & [bc^2] & & [c^3] \end{array}$$

To further illustrate the point, here are some common inequalities:

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & -1 & & -1 & & 1 \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & -1 & & -1 & \\ 1 & & -1 & & 1 \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & -1 & & -1 & & \\ & -1 & & 3 & & -1 & \\ 1 & & -1 & & -1 & & 1 \end{array}$$

$$\begin{array}{ccccc} & & 1 & & \\ & -2 & & 0 & \\ 1 & & 0 & & 0 \end{array} \qquad \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & -3 & & -3 & & & \\ & & 4 & & 0 & & 4 & & \\ & -3 & & 0 & & 0 & & -3 & \\ 2 & & -3 & & 4 & & -3 & & 2 \end{array} \qquad \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & -1 & & -1 & & & & \\ & & & 0 & & 1 & & 0 & & & \\ & & 0 & & 0 & & 0 & & 0 & & \\ & -1 & & 1 & & 0 & & 1 & & -1 & \\ 1 & & -1 & & 0 & & 0 & & -1 & & 1 \end{array}$$

First row $b^3+c^3 \geq bc(b+c), a^2+b^2+c^2 \geq ab+bc+ca$ and 3rd degree Schur.
Second row: $(a-b)^2 \geq 0$, our first example, and 5th degree Schur.

What makes this notation powerful? Once one recognizes Schur and AM-GM, then it is visually very easy to see how strong1 the inequality is. Furthermore, if one is using $S.O.S$, then they can pull out things like $\{1,-2,1\}$ and $\{1,-1,-1,1\}$ from rows (things like the upper-left diagram). One may also recognize $\{1,-4,6-4,1\}$ as $(a-b)^4$.

Finally, one can often use the identity $a^3+b^3+c^3-3abc = \frac{1}{2}\sum (a+b+c)(b-c)^2$ to convert a 3- variable application of AM-GM to $S.O.S$ form. But it is usually sufficient to just add/subtract things which are parallel to the sides of the triangle.

Example 4.2 *Let a,b,c be positive real numbers such that $a+b+c=3$. Show that :*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 1+2\sqrt{\frac{a^2+b^2+c^2}{3abc}}$$

Proof.
Homogenizing and rearranging, this becomes

$$((a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})-3)^2 \geq 4(\frac{(a+b+c)(a^2+b^2+c^2)}{abc})$$

Expanding and clearing denominators, this reduces to

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$$\sum_{sym} a^4b^2 + 2\sum_{cyc} a^3b^3 + 6a^2b^2c^2 \geq \sum_{cyc} a^4bc + 2\sum_{sym} a^3b^2c$$

Trying to fight with this mess is no easy task. But watch how easy the problem once we put in a triangle:

0
0 0
1 -2 1
2 -2 -2 2
1 -2 6 -2 1
0 -2 -2 -2 -2 0
0 0 1 2 1 0 0

It's hard to see without the triangle, but now we notice that third-degree Schur appears upside-down! The 1, −2, 1 on each side also die off: so this problem is merely the identity

$$\sum_{cyc} a^4(b-c)^2 + 2\sum_{cyc} ab(ab-bc)(ab-ac) \geq 0$$

which is evident.[4]

5 Lagrange Murderpliers

Theorem 5.1 *Let U be an open subset of R^n , and let $f: U \to R$ and $g: U \to R$ be continuous functions with continuous first derivatives. Define the constraint set*

S= { X U | g(x)=c) }

for some real numbers.

Suppose $x \in S$ is a local extrema of f along S, meaning x gives minimal or maximal value of f among points in S for some neighborhood around x. Then either $\nabla\, g(x) = 0$ at this point, or for some real number λ.

Lemma 5.2 *Let K be a compact set and $f: K \to R$ be a continuous function. Then f has a global maximum over K – there exists a point $x \in K$ such that $f(x) \leq f(y)$ for any other point $y \in K$.*

Lemma 5.3 *If $g: R^n \to R$ is a continuous function, then the set $C = \{X \in R^n \mid g(x) = C\}$ is a closed set for any real number c.*

Proof. Hence, we can escape using the following method. Let's take our function $f: U \to R$ with U open, and consider the closure $\overline{U}$ of U. Let's instead optimize $f: \overline{U} \to R^n$, where $\overline{U}$ is the closure of U. Define

$$C = \{x \in R^n | g(x) = c\}$$

Note that C is closed by our lemma. Also, define

$$S = U \cap C = \{x \in U | g(x) = c\}$$
$$\overline{S} = \overline{U} \cap C = \{x \in \overline{U} | g(x) = c\}$$

As the notation suggests, $\overline{S}$ is the closure of S, and in any case is clearly itself closed. If either of C or $\overline{U}$ is bounded, then so is $\overline{S}$, meaning $\overline{S}$ is compact! When that occurs, f achieves a maximum $x \in \overline{S}$ somewhere on the set $C \cap \overline{U}$

We can then consider two cases.

• if $x \in \overline{U} - U$, that is, x lies in the "boundary", then we can check this case manually.

• If $x \in U$, that is, x lies in the "interior", then we can use Lagrange multipliers. After all, if x is a global maximum, then it is certainly a local one as well. Here the constraint set is simply $S = C \cap U$.

Example 5.4 *For positive reals a, b, c, prove that $a + b + c \geq 3\sqrt[3]{abc}$*

Proof. The first step is de-homogenizing the inequality to assume that $a + b + c = 3$, in which case we wish to prove $abc \geq 1$. Thus, define

$$f(a, b, c) = abc$$

and

$$g(a, b, c) = a + b + c$$

Note that $\triangle g = \lambda \prec 1,1,1 \succ \neq 0$ at all points, and moreover that f and g are continuous with continuous partial derivatives.We will let $U = (0,3)^3$, $\overline{U} = [0,3]^3$ (meaning we will actually prove the inequality for all $a, b, c \leq 0$). Note that $\overline{U}$ is bounded, so the set

$$\overline{S} = \{x \in \overline{U}: g(x) = 3\}$$

is closed and hence compact. So f achieves a maximum value on $\overline{S}$ say X.

If x lies on the boundary, then at least one component of x is zero, whence $abc = 0$; this is clearly not a maximum.Otherwise, let $x = (a, b, c)$ be a point of $S = \{x \in U: g(x) = 3\}$.By Lagrange multipliers, we know

$$\nabla f(a, b, c) = \lambda \nabla g(a, b, c)$$

or

$$\prec bc, ca, ab \succ = \lambda \prec 1,1,1 \succ$$

That is, $bc = ca = ab = \lambda$. If $\lambda = 0$, then $a = b = c = 0$, and f is zero at such points. Otherwise, we conclude that $a = b = c$. Since $a + b + c = 3$, we must have $a = b = c = 1$. Hence $f(1,1,1) = 1$We find that the critical point yielding the largest value on $\overline{S}$ is (1,1,1), which gives $f(1,1,1) = 1$. This must be a maximum, and hence $f(a, b, c) \leq 1$ for all $a + b + c = 3, 0 \leq a, b, c \leq 3$, as desired.[3]

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